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The local stationary presentation of the alternating groups and the normal form

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Abstract

We give the canonical normal form for the elements of the finite or infinite alternating groups using local stationary presentation of these groups.

Keywords: Alternating group; Local group; Normal form

1. The problem

The set of finite or infinite generators $x_1, \dots, x_n$ ($n \in \mathbb{N}$ or infinite) of a group G is called the set *of local generators of depth* k if the following relations take place:

$$x_i \cdot x_j = x_j \cdot x_i,$$

when $|i - j| > k - 1$ ($k = 1$ means that G is abelian). If the set of relations between the elements $x_i, \dots, x_{i+k-1}$ does not depend on i (which means that the map sending $x_j \to x_{j+i-1}$, $j = 1, \dots, k$ is an isomorphism between the subgroup generated by $x_1, \dots, x_k$ and the subgroup

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generated by $x_i, \dots, x_{i+k-1}$), we call the set of generators *stationary*. If this is true for all $i > i_0$ for some i_0 we call it *eventually stationary*; see [6–10]. We will say in these cases that *the group has a local, stationary presentation*. The classical examples of the set of local generators and local presentations for groups are Coxeter generators and presentations of the finite Coxeter groups of infinite series, standard generators of braid groups [4], Curtis–Steinberg–Tits generators for classical matrix groups over finite fields [1, Section 4.1], etc.

The general problem, which appeared, is to describe the finite groups which have the local stationary or eventually stationary generators, to find such generators for given group if exists, to find minimal possible depth k, etc. The problem could be considered in a wider context; for example for other parametrization of the generators that intervals of integers. Namely, instead of positive integers we can consider the numeration of the generators by integers $\mathbb{Z}$, or lattice $\mathbb{Z}^d$, or semilattice $\mathbb{N}^d$ or its intervals, etc. The most interesting here are infinite locally finitely generated groups with infinite stationary set of local generators numerated by the elements of a countable group or semigroup; for example infinite symmetric groups $S_{\mathbb{N}}$.

In this paper we give the set of local stationary generators of depth 3 *for the alternating group* A_n similar to the set of depth 2 of classical Coxeter generators for symmetric groups. The well-known classical presentations of the alternating groups (see [3,4]) are very different and do not satisfy locality conditions. In order to prove the main theorem *we give the canonical normal form of the elements of the alternating group* with respect to those generators.

The result may have several possible applications. First of all it opens the way of construction of the representation theory of alternating groups in the spirit of Okounkov–Vershik approach [6, 10] independently of the representation theory of the symmetric groups; this means—to define the analogues of Gelfand–Zetlin algebra, Young–Jucys–Murphy elements, etc. Secondly, our presentation of the alternating groups allows to give various generalizations of alternating groups like anticommuting ($\mathbb{Z}_2$ or fermionic version (see [11])), q-analogs (see also [5][3]).

2. The results

Consider the group generated by the generators $x_1, \dots, x_{n-2}$ subject to the following relations:

$$x_i^3 = 1, \quad i = 1, \dots, n-2, \tag{1}$$

$$(x_i \cdot x_{i+1})^2 = 1, \quad i = 1, \dots, n-3, \tag{2}$$

$$x_i \cdot x_j = x_j \cdot x_i, \quad i, j = 1, \dots, n-2,\ |i-j| > 2, \tag{3}$$

$$x_i \cdot x_{i+1}^{-1} \cdot x_{i+2} = x_{i+2} \cdot x_i, \quad i = 1, \dots, n-4; \tag{4}$$

here n is either an integer greater than 1 or infinity. We denote the free group with these relations by S_n^+, $n \in \mathbb{N}$ (or S_∞^+).

An equivalent form of the relation (4) is the following:

$$x_{i+1} = \left[x_{i+2}, x_i^{-1}\right] = x_{i+2} \cdot x_i^{-1} \cdot x_{i+2}^{-1} \cdot x_i. \tag{5}$$

[3] We are grateful to the reviewer for this reference.

Theorem 1. *Let $n \geqslant 5$ be an integer. Then $S_n^+ \simeq A_n$. In another words the relations* (1)–(4) *define the stationary local presentation of the alternating group*:

$$A_n = S_n^+ = \langle x_1, \ldots, x_{n-2} \mid (1)\text{–}(4) \rangle, \quad n = 5, \ldots; \qquad A_4 = S_4^+ = \langle x_1, x_2 \mid (1), (2) \rangle.$$

The cases $n = 2, 3$ are trivial.

The next theorem shows that the set of relations can be reduced—the relations are not independent:

Theorem 2. *Relations* (4) *for all $i \geqslant 2$ follow from all relations* (1)–(3) *and relation* (4) *for $i = 1$ (i.e., $x_1 \cdot x_2^{-1} \cdot x_3 = x_3 \cdot x_1$).*

The key step in the proof of Theorem 1 will be a normal form for the elements of S_n^+ which is of independent interest.

We introduce the following notation.

Definition 1. For $m = 1, 2, \ldots,$ and $j = 0, 1, \ldots, m + 1,$ we introduce the following elements of S_m^+:

$$\begin{aligned} y_{m,0} &= x_m \cdot x_{m-1} \cdots x_2 \cdot x_1^2, \\ y_{m,k} &= x_m \cdot x_{m-1} \cdots x_k, \quad k = 1, \ldots, m, \\ y_{m,m+1} &= \mathrm{id}. \end{aligned}$$

In particular, $y_{1,0} = x_1^2$, $y_{1,1} = x_1$, $y_{1,2} = \mathrm{id}$.

Theorem 3 *(Normal form). For each $n \geqslant 3$ and each element $X \in S_n^+$ there exist integers $k_1, \ldots, k_{n-2}$ such that $0 \leqslant k_j \leqslant j + 1$, $j = 1, \ldots, n - 2$ and the element X has a representation of the form*:

$$X = y_{1,k_1} \cdot y_{2,k_2} \cdot y_{n-2,k_{n-2}}. \tag{6}$$

In particular, x_{n-2} appears in that form at most once, and the generator x_{n-k} appears at most $k - 1$ times, $k = 2, \ldots, n - 1$.

There is a homomorphism from the group S_n^+ (as an abstract group) to the alternating group A_n defined as follows: $x_i \mapsto (i, i + 1, i + 2)$, $i = 1, \ldots, n - 2$. Theorem 1 claims that this homomorphism is the isomorphism of the groups S_n^+ and A_n. It is not difficult to prove that in the alternating group A_n for all $n \neq 6$, $n > 2$ this is a unique solution (up to conjugacy) of the system of relations above, and two solutions in the case A_6.[4] Combining Theorem 3 with Theorem 1 we obtain the canonical normal form for the elements of alternating groups with respect to those generators.

[4] For symmetric group S_n the system of the Coxeter relations also has unique solution up to conjugacy for all $n \neq 6$; in S_6 there are two non-conjugacy solutions of those relations. The reason of that in both cases is that $Out(S_6) = Aut(S_6)/Inn(S_6) = \mathbb{Z}/2$; for $n \neq 6 - Aut(S_n) = Inn(S_n)$.

3. Proofs

We start with the proof of Theorem 3 about normal form in the group S_n^+. Then we use it for the proof of Theorem 1.

Proof of Theorem 3. The proof of the theorem is similar to the deduction of the canonical form for the elements of the symmetric group as Coxeter group. It goes by induction on n with the base $n = 3, 4$.

We will use the following transformation rules, which are consequences of relations (1)–(4):

$$x_{i+1}^2 = x_{i+1}^{-1} = x_i \cdot x_{i+1} \cdot x_i, \quad i = 1, 2, \ldots, n-3 \text{ (relation 2)}, \tag{7}$$

$$x_{i+2} \cdot x_i = x_i \cdot x_{i+1}^{-1} \cdot x_{i+2}, \quad i = 1, 2, \ldots, n-4 \text{ (relation 4)}, \tag{8}$$

$$x_j \cdot x_i = x_i \cdot x_j, \quad j - i \geqslant 3 \text{ (relation 3)}. \tag{9}$$

$n = 3$. In that case any element of S_3^+ can be written in a unique way as was done above:

$$x_1 = y_{1,1}, \qquad x_1^2 = y_{1,0}, \qquad \mathrm{id} = y_{1,2}.$$

So, this is the group of order 3 isomorphic to A_3.

$n = 4$. Using (1) we can write X as a word in x_1, x_1^2, x_2, x_2^2. We replace each occurrence of x_2^2 by $x_1 \cdot x_2 \cdot x_1$ using transformation rule (7) for $i = 1$. Thus, we may assume that x_2^2 does not appear in X. Next, any substring $x_2 \cdot x_1 \cdot x_2$ can be replaced by $x_1^2 = x_1^{-1}$ and any substring $x_2 \cdot x_1^2 \cdot x_2$ can be transformed into

$$x_2 \cdot x_1 \cdot x_1 \cdot x_2 = x_1^{-1} \cdot x_2^{-1} \cdot x_2^{-1} \cdot x_1^{-1} = x_1^2 \cdot x_2 \cdot x_1^2$$

(both transformations above are consequences of (2) and (1)). Thus, we may assume that x_2 appears at most once. In other words, any X can be written in the form

$$x_1^a, \quad a = 0, 1, 2;$$

(this is the subgroup $\mathbb{Z}/3 \simeq A_3$) or

$$x_1^a x_2 x_1^b, \qquad \text{where } a, b \in \{0, 1, 2\},$$

which gives another 9 elements, so we have 12 elements of $S_4^+ \simeq A_4$.

Inductive step. Assume the claim is true for $n \geqslant 4$, we prove it for $n + 1$.

Clearly, the subgroup of S_{n+1}^+ generated by the first $n - 2$ letters $x_1, \ldots, x_{n-2}$ is a quotient of S_n^+ (actually, as we see later, they are isomorphic, but we do not use this fact here; see Corollary 2). In particular, by the inductive hypothesis any word in $x_1, \ldots, x_{n-2}$ can be reduced to its normal form (6).

First of all, each element $X \in S_{n+1}^+$ can be represented in the form:

$$X = X_1 \cdot x_{n-1}^{\alpha_1} \cdot X_2 \cdot x_{n-1}^{\alpha_2} \cdots x_{n-1}^{\alpha_m} \cdot X_{m+1},$$

where $m \geqslant 0$, $\alpha_j \in \{1, 2\}$, $j = 1, \ldots, m$, and X_j, $j = 1, \ldots, m+1$ are words in $x_1, \ldots, x_{n-2}$. Because $n \geqslant 4$, we have the relation

$$x_{n-1}^2 \overset{\text{by (1)}}{=} x_{n-1}^{-1} \overset{\text{by (2)}}{=} x_{n-2} \cdot x_{n-1} \cdot x_{n-2}. \tag{10}$$

Therefore, it is enough to consider the case where $\alpha_j = 1$ for all $j = 1, \ldots, m$:

$$X = X_1' \cdot x_{n-1} \cdot X_2' \cdot x_{n-1} \cdots x_{n-1} \cdot X_{m+1}'. \tag{11}$$

Assume that $m \geqslant 2$. We show that we can transform the right-hand side of (11) into another word with fewer occurrences of x_{n-1}. Consider the fragment of the above word between two consecutive generators x_{n-1}:

$$X = \cdots x_{n-1} \cdot X_j' \cdot x_{n-1} \cdots.$$

By the inductive hypothesis X_j' can be written in the normal form (6). In particular, x_{n-2} appears in that normal form at most once. Using transformation rules (8)–(9) we can shift x_{n-1} at the left-hand side until we reach x_{n-2} (if any) or the next x_{n-1}. In the latter case we have x_{n-1}^2 and use (10) to diminish the number of x_{n-1}'s. In the former case we obtain

$$X = \cdots x_{n-1} \cdot y_{n-2,k} \cdot x_{n-1} \cdots$$

for some k, $0 \leqslant k \leqslant n-1$. Now, using (3) we shift x_{n-1} at the right-hand side to the left until we reach x_{n-3} (if any) or x_{n-2}. There are small differences in further analysis for $n = 4$ and $n > 4$. First assume that $n > 4$. The above transformations lead us to one of the two following substrings:

$x_{n-1} \cdot x_{n-2} \cdot x_{n-1}$, which is equal to x_{n-2}^2 by (7), or $x_{n-1} \cdot x_{n-2} \cdot x_{n-3} \cdot x_{n-1}$, which is equal to

$$\begin{aligned}
x_{n-1} \cdot x_{n-2} \cdot x_{n-3} \cdot x_{n-1} &\overset{\text{by (2)}}{=} x_{n-2}^{-1} \cdot x_{n-1}^{-1} \cdot x_{n-3} \cdot x_{n-1} \\
&\overset{\text{by (4)}}{=} x_{n-2}^{-1} \cdot x_{n-3} \cdot x_{n-1}^{-1} \cdot x_{n-2} \cdot x_{n-1} \\
&\overset{\text{by (2)}}{=} x_{n-2}^{-1} \cdot x_{n-3} \cdot x_{n-1}^{-2} \cdot x_{n-2}^{-1} \\
&\overset{\text{by (1)}}{=} x_{n-2}^{2} \cdot x_{n-3} \cdot x_{n-1} \cdot x_{n-2}^{2}.
\end{aligned}$$

In both cases we diminish the number of x_{n-1}'s.

If $n = 4$, then there is one more case to be considered, namely, the substring $x_3 \cdot x_2 \cdot x_1^2 \cdot x_3$. We have

$$x_3 \cdot x_2 \cdot x_1^2 \cdot x_3 \overset{\text{by (4)}}{=} x_3 \cdot x_3 \cdot x_1^{-1} \cdot x_3^{-1} \cdot x_1 \cdot x_1^2 \cdot x_3 \overset{\text{by (1)}}{=} x_3^2 \cdot x_1^{-1} \overset{\text{by (10)}}{=} x_2 \cdot x_3 \cdot x_2 \cdot x_1^{-1},$$

again diminishing the number of x_3's.

Thus, it is enough to consider (11) with $m \leqslant 1$.

If $m = 0$, i.e., x_{n-1} does not occur in X, we may apply the inductive hypothesis and obtain the normal form

$$X = y_{1,k_1} \cdot y_{2,k_2} \cdots y_{n-2,k_{n-2}} \cdot \mathrm{id} = y_{1,k_1} \cdot y_{2,k_2} \cdots y_{n-2,k_{n-2}} \cdot y_{n-1,n}.$$

If $m = 1$, i.e., $X = X_1' \cdot x_{n-1} \cdot X_2'$, we apply the inductive hypothesis to X_2' and write

$$X = X_1' \cdot x_{n-1} \cdot y_{1,j_1} \cdots y_{n-2,j_{n-2}}.$$

Using transformation rules (8)–(9) we shift x_{n-2} to the right until we reach $y_{n-2,j_{n-2}}$. Therefore,

$$X = X_1'' \cdot x_{n-1} \cdot y_{n-2,j_{n-2}} = X_1'' \cdot y_{n-1,j_{n-2}}.$$

Applying the inductive hypothesis to X_1'' we have

$$X = y_{1,k_1} \cdot y_{2,k_2} \cdots y_{n-2,k_{n-2}} \cdot y_{n-1,j_{n-2}}$$

thus completing the proof. □

Lemma 1. *Consider the following elements of A_n: $x_i = (i, i+1, i+2)$, $i = 1, \dots, n-2$. The relations* (1)–(4) *are true for these elements.*

This fact can be checked directly.

Thus, the group A_n is a factor-group of the group S_n^+. To prove Theorem 1 it is enough to find the order of the group S_n^+, namely, to prove that it contains at most $\frac{1}{2}n!$ elements.

Proof of Theorem 1. By Theorem 3 the order of S_n^+ is at most $\frac{1}{2}n!$. Since S_n^+ projects onto A_n by Lemma 1, we conclude that $S_n^+ \simeq A_n$. □

In particular, this implies that for any $X \in S_n^+ \simeq A_n$ the normal form of the shape (6) is unique.

Proof of Theorem 2. It follows by induction from the calculations below. Suppose that relation (4) (or its equivalent form (5)) is true for some i. We prove it for $i + 1$. Namely,

$$\begin{aligned}
x_{i+3} \cdot x_{i+1}^{-1} \cdot x_{i+3}^{-1} \cdot x_{i+1} \cdot x_{i+2}^{-1} &\overset{\text{by (1)}}{=} x_{i+3} \cdot x_{i+1}^{-1} \cdot x_{i+3}^{-1} \cdot x_{i+1} \cdot x_{i+2} \cdot x_{i+2} \\
&\overset{\text{by (2)}}{=} x_{i+3} \cdot x_{i+1}^{-1} \cdot x_{i+3}^{-1} \cdot x_{i+2}^{-1} \cdot x_{i+1}^{-1} \cdot x_{i+2} \\
&\overset{\text{by (2)}}{=} x_{i+3} \cdot x_{i+1}^{-1} \cdot x_{i+2} \cdot x_{i+3} \cdot x_{i+1}^{-1} \cdot x_{i+2} \\
&\overset{\text{by (5)}}{=} x_{i+3} \cdot x_i^{-1} \cdot x_{i+2} \cdot x_i \cdot x_{i+3} \cdot x_i^{-1} \cdot x_{i+2} \cdot x_i \\
&\overset{\text{by (3)}}{=} x_{i+3} \cdot x_i^{-1} \cdot x_{i+2} \cdot x_{i+3} \cdot x_{i+2} \cdot x_i \\
&\overset{\text{by (3)}}{=} x_i^{-1} \cdot x_{i+3} \cdot x_{i+2} \cdot x_{i+3} \cdot x_{i+2} \cdot x_i \\
&\overset{\text{by (2)}}{=} x_i^{-1} \cdot x_i = 1. \qquad \square
\end{aligned}$$

We give two important corollaries of Theorems 1 and 3. Since the order of $S_n^+ = \frac{1}{2}n!$ we immediately have

Corollary 1. *For each element $X \in S_n^+$ its normal form* (6) *is unique.*

Corollary 2. *The subgroup of S_{n+1}^+ generated by the first $n-2$ generators $x_1, \dots, x_{n-2}$ is isomorphic to S_n^+.*

Proof. Let H be the subgroup of S_{n+1}^+ generated $x_1, \dots, x_{n-2}$. Clearly, it is a factor of S_n^+. It follows from the proof of Theorem 3, that H has index $n+1$ in S_{n+1}^+. By order considerations, H must be isomorphic to S_n^+ and we can identify these two groups. $\square$

Theorems 1 and 3 give the way for the construction of the whole theory for alternating groups independently from symmetric groups—Bruhat order, Gelfand–Tsetlin algebra and so on.

4. The classical generators

We conclude the paper by relating our presentation of the group A_n with the well-known one studied by Carmichael [3], see modern explanation in [4]:

$$A_n \cong \langle v_1, \dots, v_{n-2}\colon\ v_i^3 = 1,\ (v_i v_j)^2 = 1,\ i, j = 1, \dots, n-2,\ i \neq j \rangle. \tag{12}$$

Let us define v_i, $i = 1, \dots, n-2$ by

$$v_i = \left(\prod_{k=i}^{n-2} x_k^{-1} \right)^{-1} \cdot \prod_{k=i+1}^{n-2} x_k^{-1} = x_{n-2} \cdots x_{i+1} \cdot x_i \cdot x_{i+1}^{-1} \cdots x_{n-2}^{-1}. \tag{13}$$

Using (1)–(4) we can find that the converse transformation is given by

$$x_i = \left(\prod_{j=i+1}^{n-2} v_j \right)^{-1} \cdot \prod_{j=i}^{n-2} v_j = v_{n-2}^{-1} \cdots v_{i+1}^{-1} \cdot v_i \cdot v_{i+1} \cdots v_{n-2}. \tag{14}$$

Thus $v_1, \dots, v_{n-2}$ also generate S_n^+. Moreover, they satisfy all the identities in presentation (12). This gives an independent proof of Carmichael's result. Vice versa, one can deduce our Theorem 1 from Carmichael's result. We leave details for the reader.

Remark. In the recent paper [2][5] presentations of the analogues of the alternating groups were given for all classical series of the Coxeter groups. In particular for the usual alternating group the authors have used a well-known (see [4]) set of generators:

$$r_1 = (2,3)(1,2) = (1,2,3), \qquad r_i = (1,2)(i+1, i+2), \quad i > 1;$$

and relations:

[5] We are grateful to professor A. Postnikov who had informed about paper [2].

$$r_1^3 = r_i^2 = 1, \quad i > 1; \qquad \left(r_1^{-1} r_2\right)^3 = (r_i r_{i+1})^3 = 1, \quad i = 2, \dots, n-2;$$
$$(r_i r_j)^2 = 1, \quad |i - j| > 1.$$

This set of the generators *is not local* because the last relation means that r_i and r_j commute only if the elements r_i and r_j are of order two, but $r_1^3 = 1 \neq r_1^2$, so $r_1 \cdot r_i \neq r_i \cdot r_1$ for all $i > 1$. It is interesting to find the sets of local generators for all groups which were considered in [2].

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